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PSEUDOMOMENTUM AND MATERIAL FORCES IN INHOMOGENEOUS MATERIALS

(APPLICATION TO THE FRACTURE OF ELECTROMAGNETIC MATERIALS IN ELECTROMAGNETOELASTIC FIELDS)†

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Abstract—The balance of pseudomomentum (covariant *material*, canonical momentum) is established in different ways for both pure finite-strain anisotropic elasticity and electromagnetoelasticity in the Galilean approximation for materially inhomogeneous solids. This balance law relates pseudomomentum, Eshelby's energy-momentum tensor, and the material inhomogeneity force. The relationship with the Hamiltonian canonical formulation of finite-strain elasticity is outlined and consequences for the evaluation of energy release rate via path-independent integrals in electro- or magnetoelasticity are drawn. This work, of a fairly general nature, builds on the pioneering results of the Stanford group around G. Herrmann.

"I spend money on ... because it is necessary, but to spend it on science, that is pleasant to me."
George III of England, 1738–1820

1. INTRODUCTION

In recent years much attention has been focused on the notion of *material forces* and their relationship with the general theory of *elastic inhomogeneities*. The latter show up in natural or artificially prepared, layered and composite structures, in media with continuous distributions of certain types of defects, the theory of brittle fracture, etc. A *material* force is, *per se*, the force which is generated, or produces a work, in a *material* displacement, whether finite or infinitesimal. As such, these forces are *not* the physical forces (essentially gravitational and electromagnetic forces) to which the classical statement of the principles of virtual work and power has accustomed us for two centuries (see Maugin, 1980; Dugas, 1955). Rather, as clearly understood in the pioneering works of Eshelby (1951, 1970, 1975), they are those fictitious forces which help one describe the stress-energy concentration at defects or, more generally, rather abrupt changes in material properties. Working in a continuum, here the material manifold $\mathcal{M}^3$, in *statics* one expects such forces to derive from a second-order (material) tensor, in fact the *Eshelby energy-momentum tensor* (Eshelby, 1951). Moreover, in the *dynamical* case, in the same way as the dynamical (un)balance or (non)conservation of *linear* momentum of Newton is a much deeper physical statement than the equilibrium of forces in statics (the so-called parallelogram of forces; see, e.g., Dugas, 1955), to know or show that the relation between material forces and Eshelby's tensor is through the *dynamical* (un)*balance* or (non)conservation‡ of so-called *pseudo-momentum* is a far-reaching advance because it fosters many direct generalizations that

† Dedicated to George Herrmann on the occasion of his 70th birthday.
‡ Parodying Lewis Carroll and his *un*birthdays, we may say that *un*balances and *non*conservations are much more frequent and fruitful than balances and conservations.

have been missed in static views. The present work establishes such a dynamical law in both pure nonlinear anisotropic finite-strain elasticity and electromagnetoelasticity. In this general (nonrelativistic) *electrodynamical* framework the true dichotomy between Maxwellian, all-pervasive, electromagnetic fields (**E** and **B**) and essentially *material* ones (electric polarization and magnetization) is clearly exhibited. From a different point of view, this gives us access to an efficient means for evaluating the influence of electromagnetic fields on fracture properties of brittle materials. The way is also paved for generalizations including *nonsimple* materials in the sense of Noll (Truesdell and Noll, 1965) and *dissipative* ones.

Many of the initial developments in the subject matter belong to the Stanford group around George Herrmann (e.g. Golebiewska-Herrmann, 1981, 1982; Pak and Herrmann, 1986a,b; Herrmann and Sosa, 1986; Eschein and Herrmann, 1987). This is an occasion to pay our tribute to this group and its leader. But most of the developments reported hereinafter are rather the result of our long and patient involvement in nonlinear continuum mechanics, its variational formulation and thermomechanical framework, and the general electrodynamics of continua (e.g. Maugin and Eringen, 1977; Maugin, 1980, 1988; Eringen and Maugin, 1990). They also lean heavily on works already published or in progress (Epstein and Maugin, 1990a,b, 1991; Maugin and Epstein, 1991; Maugin, 1990, 1991; Maugin and Trimarco, 1991a; in press).

Section 2 recalls the results of the purely elastic case. Section 3 introduces material electromagnetic fields. Section 4 presents the balance of pseudomomentum for electromagnetic solids. Section 5 relates to the consequences of the former for the brittle fracture of piezoelectric ceramics or magnetostrictive ferromagnets. Comments and conclusions are given in Section 6.

2. NOTATION, PURELY ELASTIC CASE

We use the standard notation of nonlinear continuum mechanics (Truesdell and Toupin, 1960; Maugin, 1988) and consider the Green–Piola energy-based theory of inhomogeneous anisotropic finite-strain elasticity (Ogden, 1984, Section 4.3). In the absence of physical body forces, in all regular points **X** of the material volume V of the material manifold $\mathcal{M}^3$ occupied by the elastic body at time t, we have the following local balance laws of (physical) linear and angular momenta:

$$0 = \mathrm{div}_R \mathbb{T} - \frac{\partial}{\partial t} \mathbf{p}_R|_{\mathbf{X}\ \mathrm{fixed}}, \quad \mathbb{F}\mathbb{T}^T = \mathbb{T}\mathbb{F}^T, \tag{1}$$

where

$$\mathbb{T} = \partial W(\mathbb{F}, \mathbf{X})/\partial \mathbb{F} = (\partial \tilde{W}(\mathbb{E}, \mathbf{X})/\partial \mathbb{E})\mathbb{F}^T, \tag{2}$$

$$\mathbf{p}_R = \rho_0(\mathbf{X})\mathbf{v}, \quad \mathbf{v} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\mathbf{X}}, \quad \mathbb{F} = \frac{\partial \boldsymbol{x}}{\partial \mathbf{X}} = (\nabla_R \boldsymbol{x})^T. \tag{3}$$

Here $\mathbf{x} = \boldsymbol{x}(\mathbf{X}, t)$ is the *direct* motion (a time-parametrized diffeomorphism of $\mathbb{R}^3$ onto itself), $\mathbb{T}$ is the first Piola–Kirchhoff stress tensor (*not* a tensor *per se* but a two-point field), $\mathbf{p}_R$ is the (contravariant) physical linear momentum per unit volume in the reference configuration K_R, $\mathbb{F}$ is the direct motion gradient (tensorial and thermodynamical dual of $\mathbb{T}$), $\mathbf{v}$ is the physical velocity (with components in the actual configuration K_t), ρ_0 is the matter density at K_R, ∇_R and div_R are *material* gradient and divergence operators, and $\mathbb{E}$ is the Lagrangian finite strain such that

$$\mathbb{E} = \tfrac{1}{2}(\mathbb{C} - \mathbb{1}_R), \quad \mathbb{C} = \mathbb{F}^T\mathbb{F}, \tag{4}$$

where $\mathbb{C}$ is the Cauchy finite strain and $\mathbb{1}_R$ is the unit dyadic on the material manifold $\mathcal{M}^3$ of "points" **X**. *Material inhomogeneity* is directly reflected in the *explicit* dependence of density ρ_0 and strain-energy function, W or $\tilde{W}$ per unit volume of K_R, on $\mathbf{X}$ (e.g. the dependence of elasticity coefficients on $\mathbf{X}$). This dependence here is assumed to be smooth so that the operations $\nabla_R \rho_0$ and $(\nabla_R W)_{\mathrm{expl}}$ (explicit gradient) are well defined. Further generalization may involve distribution-like concepts and jumps in ρ_0 and W as functions of X at specific surfaces, lines or points in $\mathcal{M}^3$, but this is not dwelt upon here. The following identities are more or less obvious:

$$\left.\frac{\partial \mathbb{F}^T}{\partial t}\right|_{\mathbf{X}} = \nabla_R \mathbf{v}, \tag{5}$$

$$(\nabla_R \mathbb{F})^T \cdot \frac{\partial W}{\partial \mathbb{F}} = \mathrm{div}_R(W \mathbb{1}_R) - (\nabla_R W)_{\mathrm{expl}}, \tag{6}$$

$$\nabla_R \cdot (J_F \mathbb{F}^{-1}) = \mathbf{0}, \quad \nabla \cdot (J_F^{-1} \mathbb{F}) = \mathbf{0}, \tag{7}$$

where

$$\mathbb{F}^{-1} = (\mathbb{F})^{-1} = \partial \tilde{\mathbf{X}} / \partial \mathbf{x} = (\nabla \tilde{\mathbf{X}})^T \tag{8}$$

if $\boldsymbol{\varkappa}^{-1} = \tilde{\mathbf{X}}(\mathbf{x}, t)$ denotes the *inverse* motion, $\mathbb{F}^{-1}$ the inverse-motion gradient, and ∇ the *spatial* gradient, with

$$J_F = \det \mathbb{F} > 0, \quad J_F^{-1} = J_{F^{-1}} = \det \mathbb{F}^{-1}. \tag{9}$$

Then we can also define a *material* velocity field $\mathbb{V}$ such that (compare to $(3)_2$ and (5))

$$\mathbb{V} = \left.\frac{\partial \tilde{\mathbf{X}}}{\partial t}\right|_{\mathbf{x}\ \mathrm{fixed}}, \quad \left.\frac{\partial (\mathbb{F}^{-1})^T}{\partial t}\right|_{\mathbf{x}} = \nabla \mathbb{V}. \tag{10}$$

Through the chain rule of differentiation one immediately shows that $\mathbf{v}$ and $\mathbb{V}$ are related by (note the *minus* sign)

$$\mathbf{v} = -\mathbb{F} \cdot \mathbb{V}, \quad \mathbb{V} = -\mathbb{F}^{-1} \cdot \mathbf{v}. \tag{11}$$

Pursuing the exploitation of $\boldsymbol{\varkappa}^{-1}$ we also write

$$\mathbb{C}^{-1} = (\mathbb{C})^{-1} = (\mathbb{F}^{-1})^T \mathbb{F}^{-1}, \tag{12}$$

the so-called *Piola* finite strain. As a matter of fact, a consistent approach to static nonlinear elasticity based on the inverse motion goes back to G. Piola in the 1840s (cf. Truesdell and Toupin, 1960). This is somewhat dual to the Cauchy–Green description in terms of the direct motion as noticed long ago by Deucker (1940–41). Then we can state the following result (Maugin, 1990; Maugin and Trimarco, 1991; submitted).

Theorem 2.1. The *material inhomogeneity* force $\mathbf{f}^{\mathrm{inh}}$ per unit volume in K_R, the *dynamic Eshelby energy-momentum tensor* $\mathbf{b}$, and the (covariant, material) *pseudomomentum* $\mathscr{P}$, at any regular point $\mathbf{X}$, satisfy jointly the (un)balance law of pseudomomentum

$$\mathbf{0} = \mathbf{f}^{\mathrm{inh}} + \mathrm{div}_R \mathbf{b} - \frac{\partial}{\partial t} \mathscr{P}|_{\mathbf{X}\ \mathrm{fixed}}, \tag{13}$$

with the definitions

$$\mathbf{f}^{inh} = (\nabla_R \mathscr{L})_{expl}, \quad \mathscr{L} = \tfrac{1}{2}\rho_0(\mathbf{X})\mathbf{v}^2 - W(\mathbb{F}, \mathbf{X}), \tag{14}$$

$$\mathbf{b} = -(\mathscr{L}\mathbb{1}_R + \mathbb{F}^T\mathbb{T}), \tag{15}$$

$$\mathscr{P} = -\mathbb{F}^T\mathbf{p}_R \equiv \rho_0(\mathbf{X})\mathbb{C}\cdot\mathbb{V} = \mathscr{P}(\mathbf{X}, t). \tag{16}$$

Proof. This is proved directly by multiplying (1) scalarly to the left by $\mathbb{F}^T$, integrating by parts, using the identities (5)–(7) and the definitions (14) and (15). Note that $(14)_2$ is a *mere* definition which coincides with that of the so-called density of Lagrangian per unit volume in K_R, and the last of (16) is trivially proved.

More astutely, we can also use the frame-indifferent (objective) energy $\tilde{W}(\mathbb{E}, \mathbf{X})$ to write $\mathbf{b}$ as

$$\mathbf{b} = [\tilde{W}(\mathbb{E}, \mathbf{X}) - \tfrac{1}{2}\rho_0(\mathbf{X})\mathscr{P}\cdot\mathbb{V}]\mathbb{1}_R - \mathbb{C}\cdot\mathbb{S} \tag{17}$$

together with

$$\mathbb{S} = \partial\tilde{W}/\partial\mathbb{E}, \quad \mathbf{b}\mathbb{C} = \mathbb{C}\mathbf{b}^T, \tag{18}$$

and $\mathbb{S}$ is the second (symmetric) Piola–Kirchhoff stress (the so-called thermodynamical stress, as it clearly is the thermodynamical dual of $\mathbb{E}$) and (18) expresses the local angular-momentum balance law in a complete covariant material form. Here $\mathbb{C}$ plays the role of *deformed metric* on $\mathscr{M}^3$. Contrary to $\mathbb{T}$, both $\mathbb{S}$ and $\mathbf{b}$ are true material tensors, the former being contravariant while $\mathbf{b}$ is mixed. Hence $(18)_2$ may also be stated as: "$\mathbf{b}$ is symmetric with respect to $\mathbb{C}$".

The following comments are in order. While $(1)_1$ is a physical balance law (still projected on the spatial frame), eqn (13) has components in local charts on $\mathscr{M}^3$. Therefore, it can be generated by a variation of $\mathbf{X}$ *at fixed current point* $\mathbf{x}$. Indeed, five methods can be envisaged to arrive at the statement (13)–(14). These are: (i) the above-given direct method; (ii) by applying Noether's theorem for $\mathbf{X}$-translations to a variational principle which primarily expresses the δ_X(at fixed $\mathbf{X}$)-variation of the Hamiltonian action

$$\mathscr{A}[\mathbf{x}] = \int_t \mathrm{d}t \int_V \mathscr{L}\,\mathrm{d}V, \tag{19a}$$

which yields first $(1)_1$—see Nelson (1979, Chap. 4); (iii) by computing directly $(\nabla_R\mathscr{L})_{expl}$ and accounting for (1)—this is performed by Eischen and Herrmann (1987) for linear elasticity or Pak (1990) and Maugin and Epstein (1991) in small and finite strain electro-elasticity, respectively; (iv) by applying a general invariance involving even nonintegrable mappings of $\mathscr{M}^3$ onto itself as done by Epstein and Maugin (1990a,b) but in statics, and (v) by direct evaluation of the δ_x(at fixed $\mathbf{x}$)-variation of the Hamiltonian action

$$\mathscr{A}'[\tilde{\mathbf{X}}, t] = \int_t \mathrm{d}t \int_{\mathscr{V}} (J_F^{-1}\mathscr{L})\,\mathrm{d}v = \mathscr{A}[\mathbf{x}^{-1}, t], \tag{19b}$$

where, obviously, $\mathscr{V}$ is the "deformed" of V by the motion, and $\mathscr{L}$ must be expressed in terms of the inverse motion, i.e.

$$\mathscr{L} = \mathscr{L}(\mathbf{X}, \mathbb{F}^{-1}, \mathbb{V}) = \tfrac{1}{2}\rho_0(\mathbf{X})\mathbb{V}\cdot\mathbb{C}\cdot\mathbb{V} - \hat{W}(\mathbb{F}^{-1}, \mathbf{X}). \tag{20}$$

This variation yields directly (cf. Maugin and Trimarco, 1991; submitted)

$$0 = \mathbf{F}^{\mathrm{inh}} + \operatorname{div}(\mathbb{B} - \mathbf{v} \otimes \hat{\mathscr{P}}) - \frac{\partial}{\partial t}\hat{\mathscr{P}}\bigg|_{\mathbf{x}\text{ fixed}}, \tag{21}$$

where

$$\mathbb{B} = -(J_F^{-1}\mathscr{L}\mathbb{F} - \mathscr{T}), \quad \mathscr{T}^T\mathbb{F}^{-1} = (\mathbb{F}^{-1})^T\mathscr{T}, \tag{22}$$

$$\mathscr{T} \stackrel{\text{def}}{=} J_F^{-1}(\partial\hat{W}/\partial\mathbb{F}^{-1})^T, \tag{23}$$

$$\hat{\mathscr{P}} = J_F^{-1}\mathscr{P} = \rho\mathbb{C}\cdot\mathbb{V}, \quad \mathbf{F}^{\mathrm{inh}} = J_F^{-1}\mathbf{f}^{\mathrm{inh}}, \tag{24}$$

where $\mathbb{B}$ and $\mathscr{T}$ bear no special names, while $\hat{\mathscr{P}}$ is the material pseudomomentum per unit volume in the *current* configuration K_t; On factorizing out J_F in eqn (21) and using (a) the relation between $(\partial/\partial t)_{\mathbf{X}\text{ fixed}}$ and $(\partial/\partial t)_{\mathbf{x}\text{ fixed}}$, and (b) the identities (7) and $(10)_2$, one is led to (13). Equation (1) then appears as the result of the application of Noether's theorem for x-translations to the action (19b). The latter formulation we may call a formulation à la Piola. Indeed, this is completed by noting that

$$\mathbb{S} = -\mathbb{C}^{-1}\cdot\mathscr{S}\cdot\mathbb{C}^{-1}, \quad \mathscr{S} = 2J_F^{-1}(\partial\bar{W}/\partial\mathbb{C}^{-1}) = \mathscr{S}^T, \quad W = \bar{W}(\mathbb{C}^{-1}, \mathbf{X}), \tag{25}$$

a formulation used in *general relativistic elastic systems* where a canonical projection onto the material manifold $\mathscr{M}^3$ is naturally preferred over a direct-motion description (Maugin, 1971, 1978) as it avoids defining a global spatial section of space-time. Furthermore, we note that

$$\delta_{\mathbf{X}}\boldsymbol{\varkappa} + \mathbb{F}\cdot\delta_{\mathbf{x}}\boldsymbol{\varkappa}^{-1} = \mathbf{0}, \tag{26}$$

but two simultaneous variations of $\boldsymbol{\varkappa}$ and $\boldsymbol{\varkappa}^{-1}$, which do not respect this constraint, can be used to generate (1) and (13) simultaneously (cf. Maugin and Trimarco, submitted; Stumpf and Le, 1990). It is not difficult to show that (21) is none other than the second of *Hamilton's canonical equations*,

$$\frac{\partial\hat{\mathscr{P}}}{\partial t}\bigg|_{\mathbf{x}\text{ fixed}} = -\frac{\delta\hat{\mathscr{H}}}{\delta\mathbf{X}}, \quad \hat{\mathscr{H}} = \hat{\mathscr{P}}\cdot\mathbb{V} - J_F^{-1}\mathscr{L}, \tag{27}$$

where $\hat{\mathscr{H}}$ is the Hamiltonian density and $\delta/\delta\mathbf{X}$ denotes the variational (functional) material gradient (Maugin and Trimarco, 1991) if $(10)_1$ is recognized as the first one:

$$\frac{\partial\tilde{\mathbf{X}}}{\partial t}\bigg|_{\mathbf{x}\text{ fixed}} = \frac{\delta\hat{\mathscr{H}}}{\delta\hat{\mathscr{P}}}. \tag{28}$$

The reader will find in other works (cf. Maugin and Trimarco, 1991; submitted) applications of (13) and (18)–(20) to the evaluation of the energy-release rate (J-integral) in brittle fracture and the conception of a fracture criterion via a variational inequality. All that needs to be noted for the moment is that the total material inhomogeneity force $\mathscr{F}^{\mathrm{inh}}$ for a control volume V, cut by a material singular surface Σ, is given by the integral

$$\mathscr{F}^{\mathrm{inh}} \stackrel{\text{def}}{=} \int_{V-\Sigma} \mathbf{f}^{\mathrm{inh}}\,\mathrm{d}V, \tag{29}$$

and this, in theory, can be evaluated by using (13) and generalized versions of Stokes' divergence and Reynolds' transport theorems (Maugin and Trimarco, 1991) even if V is mobile with respect to Σ (note the inversion of the "thought process" as it is Σ which is fixed in this description!).

3. MATERIAL ELECTROMAGNETIC FIELDS

Maxwell's equations in a magnetized and electrically polarized material are usually expressed in a fixed frame, R_L, called the Laboratory frame and, by relativists and those interested in dynamics, in a *co-moving* frame, $R_C(\mathbf{x}, t)$ at time t (see Maugin, 1988, Chap. 3; Eringen and Maugin, 1990, Chap. 3). They can as well be formulated entirely on the material manifold $\mathcal{M}^3$ as shown initially by Walker *et al.* (1965) and McCarthy (1968)—see also Lax and Nelson (1976) and others (Nelson, 1979; Maugin, 1981; Ani and Maugin, 1989). The clues to a good *material* formulation of Maxwell's equations (*in matter*, but this is a Zen-like remark) are (i) to notice that magnetic induction, electric displacement and electric polarization are *contravariant* vectors (of which the first is axial) to which divergence operators apply, while electric and magnetic fields and magnetization are *covariant* vectors (of which the last two are axial) to which one normally applies the curl operation. Furthermore the last three fields must be considered in a co-moving frame to start with (cf. Maugin, 1988, Chap. 3). Let $\mathbf{B}$, $\mathbf{D}$, $\mathbf{P}$, $\mathbf{E}$, $\mathbf{H}$ be the first five fields in R_L, and $\mathcal{M}$ be the magnetization per unit volume in $R_C(\mathbf{x}, t)$. Then in a Galilean approximation one introduces the following *material* electromagnetic fields (c is the velocity of light in vacuo)

$$\mathfrak{B} = J_F \mathbb{F}^{-1} \cdot \mathbf{B}, \quad \mathfrak{D} = J_F \mathbb{F}^{-1} \cdot \mathbf{D}, \quad \Pi = J_F \mathbb{F}^{-1} \cdot \mathbf{P},$$

$$\mathbb{M} = \mathcal{M} \cdot \mathbb{F}, \quad \mathfrak{E} = \mathbf{E} \cdot \mathbb{F} - \frac{1}{c} \mathbb{V} \times \mathfrak{B}, \quad \mathfrak{H} = \mathbf{H} \cdot \mathbb{F} + \frac{1}{c} \mathbb{V} \times \mathfrak{D}. \tag{30}$$

Then the classical (current-configuration) formulation of Maxwell's equations in non-dissipative, charge-free matter, i.e. (with Lorentz-Heaviside units)

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = 0, \quad \nabla \cdot \mathbf{D} = 0,$$

$$\mathbf{H} = \mathbf{B} - \mathbf{M}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}, \quad \mathcal{M} = \mathbf{M} + \frac{1}{c}\mathbf{v} \times \mathbf{P} \tag{31}$$

transforms to the material framework as

$$\nabla_R \times \mathfrak{E} + \frac{1}{c}\left.\frac{\partial \mathfrak{B}}{\partial t}\right|_{\mathbf{X}} = 0, \quad \nabla_R \cdot \mathfrak{B} = 0,$$

$$\nabla_R \times \mathfrak{H} - \frac{1}{c}\left.\frac{\partial \mathfrak{D}}{\partial t}\right|_{\mathbf{X}} = 0, \quad \nabla_R \cdot \mathfrak{D} = 0,$$

$$\mathfrak{H} = J_F^{-1}\mathbb{C} \cdot \bar{\mathfrak{B}} - \mathbb{M}, \quad \mathfrak{D} = J_F \mathbb{C}^{-1} \cdot \bar{\mathfrak{E}} + \Pi, \tag{32}$$

where

$$\bar{\mathfrak{E}} = \mathfrak{E} + \frac{1}{c}\mathbb{V} \times \mathfrak{B} = \mathbf{E} \cdot \mathbb{F}, \quad \bar{\mathfrak{B}} = \mathfrak{B} - \frac{1}{c}\mathbb{V} \times \bar{\mathfrak{E}} = J_F \mathbb{F}^{-1} \cdot \left(\mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E}\right). \tag{33}$$

Clearly, from $(3.2)_{1,2}$ there follows the existence of potentials φ and $\mathfrak{A}$ (in the material description; cf. Nelson, 1979, pp. 406–407) such that

$$\mathfrak{E} = \left(\nabla_R \varphi + \frac{1}{c}\frac{\partial}{\partial t}\mathfrak{A}|_X\right) \quad \mathfrak{B} = \nabla_R \times \mathfrak{A}. \tag{34}$$

Let

$$\mathscr{L}^0_{em}(\mathbf{E}, \mathbf{B}) = \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{35}$$

be the Lagrangian of free (Maxwell) fields either in a vacuum or in a body per unit volume of the current configuration K_t. At points where this expression is meaningful, we have:

Lemma 3.1. There holds the identity

$$0 = \nabla_R \mathscr{L}_{em} - \mathrm{div}_R[\mathbb{F}^T \cdot (\partial \mathscr{L}_{em}/\partial \mathbb{F})] + \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}_{em}}{\partial \mathbb{V}}\right), \quad \mathscr{L}_{em} = J_F \mathscr{L}^0_{em}. \tag{36}$$

Proof. This is proved by direct computation of $\nabla_R \mathscr{L}_{em}$ (see Maugin and Epstein, 1991, for the special case of electroelastics, but the general case is treated similarly).

At those points X we can also write, on account of (30) and (33),

$$\mathscr{L}_{em}(\mathfrak{E}, \mathfrak{B}; \mathbb{C}) = \tfrac{1}{2}J_F \bar{\mathfrak{E}} \cdot \mathbb{C}^{-1} \cdot \bar{\mathfrak{E}} - \tfrac{1}{2}J_F^{-1}\mathfrak{B} \cdot \mathbb{C} \cdot \mathfrak{B}. \tag{37}$$

To account for material inertia, elasticity and electromagnetic interactions between matter and the Maxwellian fields, we must complement (37) with the following general "matter-plus-interactions" Lagrangian per unit volume in K_R:

$$\mathscr{L}_M(\mathbf{v}, \mathbb{F}, \mathfrak{E}, \mathfrak{B}; \mathbf{X}) = \tfrac{1}{2}\rho_0 \mathbf{v}^2 - W(\mathbb{F}, \bar{\mathfrak{E}}, \mathfrak{B}; \mathbf{X}) \tag{38a}$$

or

$$\hat{\mathscr{L}}_M(\mathbb{V}, \mathbb{F}^{-1}, \mathfrak{E}, \mathfrak{B}; \mathbf{X}) = \tfrac{1}{2}\rho_0(\mathbf{X})\mathbb{V} \cdot \mathbb{C} \cdot \mathbb{V} - W(\mathbb{F}^{-1}, \bar{\mathfrak{E}}, \mathfrak{B}; \mathbf{X}). \tag{38b}$$

4. BALANCE OF PSEUDOMOMENTUM IN ELECTROMAGNETOELASTICITY

The algebra of the following developments is much facilitated by the fact that many of the intermediate computations were, in effect, carried out by Nelson (1979) in his monograph, although for a different purpose. We shall *not* repeat his proofs. Some of our results have been partially enunciated in short notes (Maugin, 1990, 1991). Here they are corrected where necessary. First we have the:

Theorem 4.1. Maxwell's equations $(32)_{3,4}$ and the balance laws of physical linear and angular momenta

$$0 = \mathrm{div}_R(\mathbb{T}^E + \mathbb{T}^F + \mathbb{P} \otimes \mathbf{p}^F/\rho) - \frac{\partial}{\partial t}(\mathbf{p}_R + \mathbf{p}^F_R), \quad \mathbb{T}^E\mathbb{F}^T = \mathbb{F}(\mathbb{T}^E)^T, \tag{39}$$

follow from the straightforward δ_X (at fixed X)-variation, accompanied by proper variations $\delta\varphi$ and $\delta\mathfrak{A}$, of the Hamiltonian action:

$$\mathscr{A}[\boldsymbol{x}, \mathbf{E}, \mathbf{B}] = \int_t \mathrm{d}t \int_V (\mathscr{L}_{em} + \mathscr{L}_M)\,\mathrm{d}V, \tag{40}$$

with the following definitions:

$$\mathbb{T}^E = \partial W/\partial \mathbb{F}, \quad \mathbb{T}^F = J_F \mathbb{F}^{-1} \mathbf{t}^F, \quad \mathbb{P} = \mathbb{F}^{-1} \cdot \mathbf{p}_R, \quad \mathbf{p}_R^F = J_F \mathbf{p}^F, \tag{41}$$

where

$$\mathbf{p}^F = \frac{1}{c} \mathbf{E} \times \mathbf{B}, \quad \mathbf{t}^F = \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B} - \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{1}. \tag{42}$$

The long proof of this shall be given elsewhere (Maugin and Trimarco, in press). Simultaneously, applying Noether's theorems for **X**-translations to the variational principle of the previous theorem, we obtain the following lemma.

Lemma 4.2. For a nondissipative, charge-free, inhomogeneous, anisotropic electromagnetoelastic body in finite transformation, the following (un)balance of pseudomomentum holds true:

$$\mathbf{0} = \mathbf{f}^{\text{inh}} + \text{div}_R \mathbf{b}' - \frac{\partial}{\partial t} \mathcal{P}'|_{\mathbf{X}\text{ fixed}}, \tag{43}$$

where $\mathbf{f}^{\text{inh}}$ is defined as in eqn (4) and the electromagnetomechanical generalizations of **b** and $\mathcal{P}$, $\mathbf{b}'$ and $\mathcal{P}'$ are given by

$$\mathbf{b}' = (W - \tfrac{1}{2}\rho_0 \mathbf{v}^2)\mathbb{1}_R - \mathbb{C} \cdot \mathbb{S}, \tag{44}$$

$$\mathcal{P}' = \mathcal{P} + \frac{1}{c} \mathbf{\Pi} \times \mathfrak{B}, \tag{45}$$

where

$$\mathbb{S} = \mathbb{S}^E - \mathbb{C}^{-1} \cdot \mathfrak{E} \otimes \mathbf{\Pi} + \mathbb{C}^{-1} \cdot \mathbb{M} \otimes \mathfrak{B}, \tag{46}$$

together with the *constitutive equations*

$$\mathbb{S}^E = \partial \tilde{W}/\partial \mathbb{E}, \quad \mathbf{\Pi} = -\partial \tilde{W}/\partial \bar{\mathfrak{E}}, \quad \mathbb{M} = \partial \tilde{W}/\partial \mathfrak{B}, \quad W = \tilde{W}(\mathbb{E}, \bar{\mathfrak{E}}, \mathfrak{B}; \mathbf{X}), \tag{47}$$

where the objectivity requirement has been applied to the strain-energy function W.

Proof. The long proof, to be given elsewhere (Maugin and Trimarco, in press), follows the one indicated by Nelson (1979, Chap. 8) in the absence of magnetization and material inhomogeneities. Essential in the inhomogeneous case is the identity (36) which has to show ultimately that the free Maxwellian fields *cannot* contribute, just by themselves, to the statement (43) and the definitions (44) and (45). That is, contrary to the physical expressions (39) which contain free-field contributions via the expressions (42)—pure free-field electromagnetic momentum and Maxwell stresses—the material expressions (44) and (45) do *not* contain such contributions. An identity of the type of (36) was missed in the original electrostatic derivation of Pak and Herrmann (1986a), resulting in the presence of a superfluous, identically vanishing, group of terms.

Remarks. (a) First we note that, formally, $\mathbf{b}'$ has the same expression as in pure elasticity except that $\mathbb{S}$ admits a canonical decomposition (46) typical of electromagnetic bodies (cf. Maugin and Eringen, 1977). (b) The result (43) with $\mathbf{f}^{\text{inh}} = \mathbf{0}$ (but with additional magnetic terms) is none other than an expression obtained by Nelson (1979, p. 159) but of which the importance was only recently discovered by that author (Nelson, 1990) in connection with the notion of *pseudo*-momentum or *canonical* momentum after the enlightening remarks of Peierls (1985). (c) The variational origin of the expression of $\mathbf{b}'$ is highlighted by noting that (44) also reads in components as

$$b_I^{\cdot J} = [\bar{W}(\mathbb{C}, \mathfrak{E}, \mathfrak{B}; \mathbf{X}) - \tfrac{1}{2}\rho_0(\mathbf{X})\mathbb{V}\cdot\mathbb{C}\cdot\mathbb{V}]\delta_I^J - 2C_{IK}\frac{\partial \bar{W}}{\partial C_{KJ}} - \mathfrak{E}_I\frac{\partial \bar{W}}{\partial \mathfrak{E}_J} - \frac{\partial \bar{W}}{\partial \mathfrak{B}^I}\mathfrak{B}^J. \quad (48)$$

(d) Both *physical* and *pseudo* electromagnetic linear momenta (in matter) are neither Minkowski's nor Abraham's proposals. In a *rigid* solid, however ($\mathbb{F} = \mathbb{1}$, $\mathbb{C} = \mathbb{1}$), the sum of our two expressions provides Abraham's momentum $(\mathbf{D} \times \mathbf{B})/c$. (e) Legendre transformations can be performed for the dependence of energy W on electromagnetic quantities (cf. Maugin, 1991). (f) In theory at least, (43) can be derived by any of the five methods mentioned in Section 2. In particular, it should be shown to be the direct consequence of the δ_x-variation of the Hamiltonian action

$$\mathscr{A}'[\tilde{\mathbf{X}}, t] = \int_t \mathrm{d}t \int_{v} J_F^{-1}(\mathscr{L}_{em} + \hat{\mathscr{L}}_M)\,\mathrm{d}v. \quad (49)$$

This, as well as the corresponding Hamiltonian canonical formulation, is left for further studies.

5. APPLICATION TO BRITTLE FRACTURE

Path-independent integrals of the J-integral type have been proposed in *electroelastostatics* by several authors (Pak and Herrmann, 1986a,b; Pak, 1990; McMeeking, 1990; Parton and Kudryavtsev, 1988; Maugin and Epstein, 1991). This can also be achieved in all generality by starting from the basic law (43). As in the purely elastic case one has to take notice of the following demonstrable theorems:

$$\int_{V-\Sigma} \frac{\partial}{\partial t}\mathscr{P}|_{\mathbf{X}}\,\mathrm{d}V = \frac{\delta}{\delta t}\int_{V-\Sigma}\mathscr{P}\,\mathrm{d}V - \int_{\partial V-\Sigma}\mathscr{P}(\mathbf{N}\cdot\mathscr{U})\,\mathrm{d}S + \int_\Sigma [\![\mathscr{P}\otimes\mathscr{U}]\!]\cdot\mathbf{N}\,\mathrm{d}S, \quad (50a)$$

$$\int_{V-\Sigma}\mathrm{div}_R\,\mathbf{b}\,\mathrm{d}V = \int_{\partial V-\Sigma}\mathbf{b}\cdot\mathbf{N}\,\mathrm{d}S - \int_\Sigma [\![\mathbf{b}]\!]\cdot\mathbf{N}\,\mathrm{d}S - \int_{\partial\Sigma}\mathrm{d}L\left[\lim_{\Gamma\to 0}\int_\Gamma \mathbf{b}\cdot\mathbf{N}\,\mathrm{d}\Gamma\right], \quad (50b)$$

where V is a regular material volume containing a flattened disk crack Σ of contour $\partial\Sigma$, and moving with spatially uniform velocity $\mathscr{U}$ with respect to Σ, Γ is an open contour in the cross-section of the torus-like tube around $\partial\Sigma$ (Fig. 1a), and $[\![\ldots]\!]$ denotes the jump of its enclosure. By letting V shrink to Σ and noting that the volume term containing the pseudomomentum will contribute zero in this limit, we obtain from (29) the following result:

$$\mathscr{F}^{\mathrm{inh}}(V-\Sigma) = \int_{\partial\Sigma}\mathbf{J}(\partial\Sigma)\,\mathrm{d}L, \quad \mathbf{J}(\partial\Sigma) = \lim_{\Gamma\to 0}\mathbf{J}_\Gamma, \quad \mathbf{J}_\Gamma = \int_\Gamma \mathbf{b}'\cdot\mathbf{N}\,\mathrm{d}\Gamma, \quad (51)$$

where $\mathbf{J}_\Gamma$ is path-independent (as readily checked). In quasi-statics, this reads

$$\mathbf{J}_\Gamma = \int_\Gamma [W\mathbf{N} - \mathbb{C}\cdot(\mathbb{S}^E\cdot\mathbf{N}) - (\nabla_R\varphi)(\Pi\cdot\mathbf{N}) - \mathbb{M}(\mathfrak{B}\cdot\mathbf{N})]\,\mathrm{d}\Gamma. \quad (52)$$

For a flat, straight, through-crack ($\partial\Sigma$ is then perpendicular to the plane of Fig. 1b), and considering the limit of small strains for the *electroelasticity of brittle ceramics*, eqns (51) and (52) produce the electroelastic J-integral by

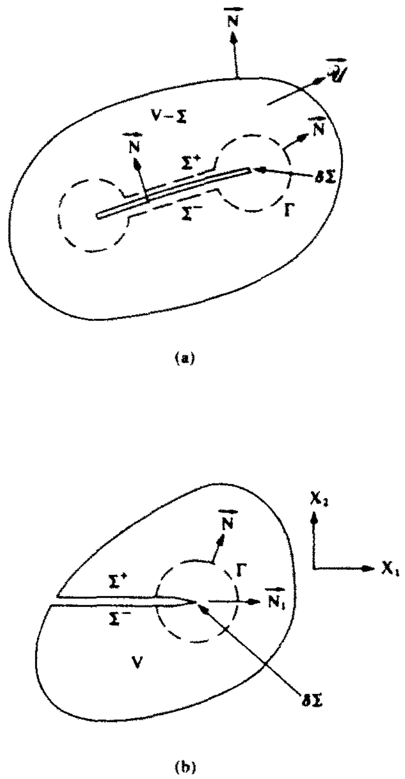

Fig. 1. (a) Disked crack in a nonlinear homogeneous elastic medium. (b) Through straight plane crack.

$$J = \lim_{\Gamma \to 0} (\mathscr{F}^{\text{inh}} \cdot \mathbf{e}_1) = \lim_{\Gamma \to 0} \int_\Gamma \left(W N_1 - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial X} - Q \frac{\partial \varphi}{\partial X} \right) \mathrm{d}\Gamma, \tag{53}$$

with

$$N_1 = \mathbf{N} \cdot \mathbf{e}_1, \quad \partial/\partial X = \mathbf{e}_1 \cdot \nabla_R, \tag{54}$$

$$\mathbf{T} = \mathbf{N} \cdot \mathbb{S}^E, \quad Q = \mathbf{N} \cdot \mathbf{\Pi}, \tag{55}$$

where $\mathbf{u}$ is the elastic displacement, $\mathbf{e}_1$ is a unit vector in the direction of extension of the crack (J may then be called the crack-extension force; see Fig. 1b), $\mathbf{T}$ is the traction vector, and Q is the surface polarization along the contour Γ. The small-strain limit for the mechanical part in passing from (52) to (53) is well explained in Casal (1978). Note that in the above approximation W is at most quadratic in the strain; however, high-order nonlinearities in pure electric properties are still allowed and electrostriction and higher-order electroelastic effects are still contained in W, $\mathbf{T}$ and Q, via $\mathbb{S}^E$ and $\mathbf{\Pi}$ (see Maugin, 1985, for these effects in ceramics and ferroelectrics). The quasi-magnetostatic case of *nonpolarized magnetostrictive brittle magnets* in a magnetic field is deduced along similar lines from (52). As we know from many works (Rice, 1968; Fletcher, 1976; Bui, 1978; Gurtin and Yatomi, 1980; Eschein and Herrmann, 1987; Maugin, in press), the J-integral is related to the *energy release rate* and this, in turn, is connected to the material *toughness* and stress intensity factors at crack tips. Thus a brittle fracture criterion can eventually be proposed which will account for electromagnetic effects (compare Maugin and Trimarco, submitted, and Stumpf and Le, 1990, for the purely mechanical case). A crack propagation criterion can be devised on the same basis but relying on a *variational inequality* rather than a variational principle (ibid.). For lack of space we shall not pursue here these exciting developments.

6. CONCLUSION: PROSPECTS

It is difficult to go further in complexity than above in the framework of *nondissipative simple* materials (those which involve only the *first* gradient $\mathbb{F}$ or $\mathbb{F}^{-1}$). However, we would like to point to three generalizations of interest if some of these restrictions are released: (i) as already noted the introduction of material gradients in the generalized function (distribution) sense will allow one to account for abrupt changes in material properties; (ii) on noting that most of the above developments (especially those using the inverse-motion description) strictly pertain to a true *field theory* (with a variational basis), higher-order material gradients can be accounted for, such as in second-grade elasticity, practically with no additional difficulty (see, e.g. Epstein and Maugin, 1991, for such extensions which will allow for continuous distributions of disclinations); and finally (iii) if the medium considered is basically elastic but also *dissipative* then the direct approach used in passing from (1) to (13) is still feasible with a more general definition of $\mathbf{f}^{inh}$ that will account for the explicit dependence of the *pseudopotential of dissipation* (see Maugin, in press, for this notion) on $\mathbf{X}$; e.g. dependence of viscosity coefficient, hardening modulus and yield stress on $\mathbf{X}$ (Maugin, in preparation). Then the notions of global inhomogeneity force $\mathscr{F}^{inh}$ and J-integral can be generalized to these nontrivial cases. This will be developed at length elsewhere.

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† This reference considers only static deformations so that there is no distinction between $(\partial/\partial t)_X$ and $(\partial/\partial t)_x$, a very unfortunate and misleading situation.